Variant 1

№ 2 Read the text

Poincaré's conjecture

(from a press release of the Clay Mathematics Institute)

In the latter part of the nineteenth century, the French mathematician Henri Poincaré was studying the problem of whether the solar system is stable. Do the planets and asteroids in the solar system continue in regular orbits for all time, or will some of them be ejected into the far reaches of the galaxy or, alternatively, crash into the sun? In this work he was led to topology, a still new kind of mathematics related to geometry, and to the study of shapes (compact manifolds) of all dimensions.

The simplest such shape was the circle, or distorted versions of it such as the ellipse or something much wilder: lay a piece of string on the table, tie one end to the other to make a loop, and then move it around at random, making sure that the string does not touch itself. The next simplest shape is the two-sphere, which we find in nature as the idealized skin of an orange, the surface of a baseball, or the surface of the earth, and which we find in Greek geometry and philosophy as the "perfect shape." Again, there are distorted versions of the shape, such as the surface of an egg, as well as still wilder objects. Both the circle and the two-sphere can be described in words or in equations as the set of points at a fixed distance from a given point (the center). Thus it makes sense to talk about the three-sphere, the four-sphere, etc. These shapes are hard to visualize, since they naturally are contained in four-dimensional space, five-dimensional space, and so on, whereas we live in three-dimensional space. Nonetheless, with mathematical training, shapes in higher-dimensional spaces can be studied just as well as shapes in dimensions two and three.

In topology, two shapes are considered the same if the points of one correspond to the points of another in a continuous way. Thus the circle, the ellipse, and the wild piece of string are considered the same. This is much like what happens in the geometry of Euclid. Suppose that one shape can be moved, without changing lengths or angles, onto another shape. Then the two shapes are considered the same (think of congruent triangles). A round, perfect two-sphere, like the surface of a ping-pong ball, is topologically the same as the surface of an egg.

In 1904 Poincaré asked whether a three-dimensional shape that satisfies the "simple connectivity test" is the same, topologically, as the ordinary round three-sphere. The round three-sphere is the set of points equidistant from a given point in four-dimensional space. His test is something that can be performed by an imaginary being who lives inside the three-dimensional shape and cannot see it from "outside." The test is that every loop in the shape can be drawn back to the point of departure without leaving the shape. This can be done for the two-sphere and the three-sphere. But it cannot be done for the surface of a doughnut, where a loop may get stuck around the hole in the doughnut.

The question raised became known as the Poincaré conjecture. Over the years, many outstanding mathematicians tried to solve it — Poincaré himself, Whitehead, Bing, Papakirioukopolos, Stallings, and others. While their efforts frequently led to the creation of significant new mathematics, each time a flaw was found in the proof. In 1961 came astonishing news. Stephen Smale, then of the University of California at Berkeley, proved that the analogue of the Poincaré conjecture was true for spheres of five or more dimensions. The higher-dimensional version of the conjecture required a more stringent version of Poincaré's test; it asks whether a so-called homotopy sphere is a true sphere. Smale's theorem was an achievement of extraordinary proportions. It did not, however, answer Poincaré's original question. The search for an answer became all the more alluring.

Smale's theorem suggested that the theory of spheres of dimensions three and four was unlike the theory of spheres in higher dimension. This notion was confirmed a decade later, when Michael Freedman, then at the University of California, San Diego, announced a proof of the Poincaré conjecture in dimension four. His work used techniques quite different from those of Smale. Freedman also gave a classification, or kind of species list, of all simply connected four-dimensional manifolds.

In the last century, there were many attempts to prove, and also to disprove, the Poincaré conjecture using the methods of topology. Around 1982, however, a new line of attack was opened. This was the Ricci flow method pioneered and developed by Richard Hamilton. It was based on a differential equation related to the one introduced by Joseph Fourier 160 years earlier to study the conduction of heat. With the Ricci flow equation, Hamilton obtained a series of spectacular results in geometry. However, progress in applying it to the conjecture eventually came to a standstill, largely because formation of singularities, akin to formation of black holes in the evolution of the cosmos, defied mathematical understanding.

Perelman's breakthrough proof of the Poincaré conjecture was made possible by a number of new elements. He achieved a complete understanding of singularity formation in Ricci flow, as well as the way parts of the shape collapse onto lower-dimensional spaces. He introduced a new quantity, the entropy, which instead of measuring disorder at the atomic level, as in the classical theory of heat exchange, measures disorder in the global geometry of the space. This new entropy, like the thermodynamic quantity, increases as time passes. Perelman also introduced a related local quantity, the *L*-functional, and he used the theories originated by Cheeger and Aleksandrov to understand limits of spaces changing under Ricci flow. Perelman deployed his new ideas and methods with great technical mastery and described the results he obtained with elegant brevity. Mathematics has been deeply enriched.

Variant 1

Read the text again and check that you have understood the main points by choosing the best answer, A, B, C or D, to these questions.

1. Which problem had motivated Poincaré to study topology?

- **A.** *The problem of finding the "perfect shape"*
- **B.** *The question of whether the planets rotating around the sun will keep roughly the same orbits forever*
- **C.** *The problem of understanding compact manifolds of arbitrary dimension*
- **D.** *An attempt of finding a new kind of mathematics similar to geometry*

2. Choose one shape that is NOT topologically the same as the surface of a ball.

- **A.** *The surface of an egg*
- **B.** *The surface of the Earth*
- **C.** *The idealized skin of an orange*
- **D.** *The surface of a tire*

3. What is the topological space that appears in the Poincaré conjecture?

- **A.** *The set of all points in the 4-space that are at a given distance from the center*
- **B.** *A four-dimensional sphere*
- **C.** *A three-dimensional torus*
- **D.** *A sphere of five or more dimensions*

4. Who proved that the analogue of the Poincaré conjecture was true for spheres of dimension four?

- **A.** *Stephen Smale*
- **B.** *Grigoriy Perelman*
- **C.** *Michael Freedman*
- **D.** *Richard Hamilton*

5. Which physical process is described by an equation similar to that of the Ricci flow?

- **A.** *The gravitational interaction in presence of black holes*
- **B.** *The evolution of the cosmos*
- **C.** *The motion of planets*
- **D.** *The way heat spreads in media*

Variant 1

№ 3 Read the article about the concept of a group. Then choose the best fragment below to fill each of the gaps.

- **A.** you're back to where you started
- **B.** simply reflect again in the same axis
- **C.** your overall movement is also a symmetry
- **D.** this operation of addition never takes you into new territory
- **E.** it's finite and instead of numbers it involves symmetries

The power of groups

(from an article by Colva Roney-Dougal in the Plus-Magazine)

Before saying what exactly a group is, let's prepare the way by looking at two examples. The first thing to think about is the collection of all whole numbers: positive, negative and zero.

You can add any two numbers to get a third, and **__1__**: when you add two whole numbers you always get another whole number as a result. The process of adding the whole number *a* to something can be reversed without leaving the realm of addition: simply add the negative of *a*, the number -*a*, and **__2__**. This nicely self-contained system is an example of a group.

The second example is a little different: **__3__**. The symmetries of a shape drawn on a piece of paper, like a square, are all those reflections and rotations that don't alter the appearance of the shape in the plane: unless you label the corners or edges, you'll never know if it's been rotated or reflected. For a square, the symmetries are the (clockwise) rotations through 90, 180 and 270 degrees about the centre point, the reflections in the horizontal, vertical and the two diagonal axes, and leaving the square fixed (not moving it at all).

Whenever you follow one symmetry by another, **__4__** because the square appears unchanged. A diagonal reflection, followed by a rotation by 90 degrees is the same as the reflection in the horizontal: combining two symmetries by doing one after the other never takes you out of the world of symmetries — just as combining two whole numbers through addition never takes you out of the world of whole numbers.

Whenever you have performed a symmetry, you can undo it again: if you have rotated clockwise through an angle θ°, simply rotate clockwise through the angle 360 - θ°, and if you have reflected in a certain axis, **__5__**. There are exactly 8 distinct symmetries of the square, and together they form a neat little self-contained system. And they do so in a very similar way as the whole numbers do when considered together with addition.

Variant 1

№4 Read the following text and then decide whether the statements following are a) true or b) false:

What is a … dimer?

(from an article by R. Kenyon and A. Okounkov in the Notices of the AMS)

A dimer is a polymer with only two atoms. A dimer covering of a graph *G* is a collection of edges that covers all the vertices exactly once, that is, each vertex is the endpoint of a unique edge. One can think of vertices of *G* as univalent atoms, each bonding to exactly one neighbor. Dimer coverings are also called perfect matchings. The dimer model is the study of natural measures (“Gibbs measures”) on the set of dimer coverings of a graph, usually a periodic planar graph such as Z^2. While not a very realistic physical model, the dimer model has intrinsic interest as an exactly solvable model that exhibits certain types of phase transitions. Since phase transitions are quite complicated phenomena in nature, any approximate model whose phase transitions can be studied analytically is valuable.

Kasteleyn, contemporaneously with Temperley and Fisher, showed how to count the number of dimer coverings of an $m \times n$ square grid, and later on any planar graph. While Kasteleyn’s result holds for any planar graph, the statement is particularly simple when *G* is a subgraph of the honeycomb graph *H* (the graph of the regular tiling of the plane by hexagons) bounded by a simple polygon (a “simply connected” subgraph). Then the number of coverings *Z* is the square root of the determinant of the adjacency matrix of *G*. The adjacency matrix *K* is the matrix indexed by the vertices of *G* defined by $K_{v,w} = 1$ or 0 according to whether *v, w* are adjacent or not. A similar statement (but with extra signs in *K*) holds for any planar graph.

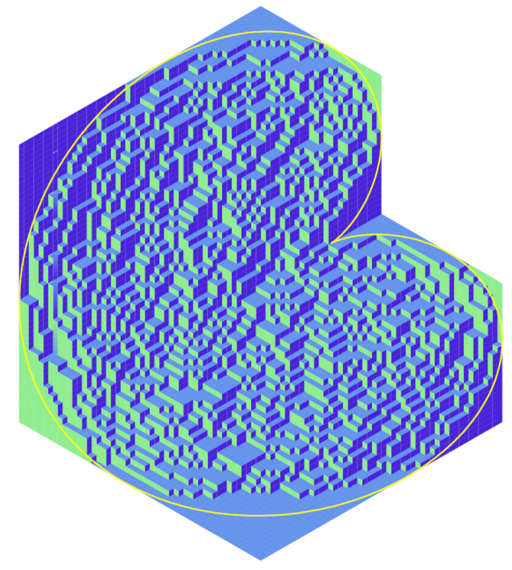

Figure 1

Dimer coverings of bipartite graphs such as the honeycomb graph or square grid can be viewed as (random) surfaces in $\mathbf{R}^2$. Concretely, dimer coverings of the honeycomb or any of its simply connected subgraphs can be represented as tilings of the plane with 60° rhombi: in this case each atom is a triangle, and dimers are obtained by gluing adjacent triangles along an edge. Clearly any such rhombus tiling can be viewed as the projection of a three-dimensional piecewise-linear surface. The function on a tiling that gives the third coordinate of this projection is called the height function. A similar kind of height function also exists for domino tilings, or dimers on any bipartite planar graph, though the definition is a little more complicated.

The tiling in Figure 1 is perfectly random, and yet one can see that there is something regular about its height function. In fact, as we take tilings of a fixed region (e.g., a polygon) with smaller and smaller rhombi, the height functions of a typical tiling will converge to some nonrandom surface called the limit shape. What is this limit shape? Figure 1 shows it has some linear pieces—those occur in the frozen regions near the corners where all tiles there are lined up the same way. How large are the fluctuations around the limit shape? What is the probability of occurrence of some fixed local pattern at a given point?
The answers to these and other questions depend on the asymptotics of the inverse Kasteleyn matrix K^{-1} as the size of the graph goes to infinity. Since K can be viewed (using work of the first author) as a discretization of the ∂-operator, it is not so surprising that complex analysis and algebraic geometry enter the scene. In the nonfrozen
region, the limit shape satisfies a nonlinear elliptic partial differential equation closely related to the complex Burgers equation $\varphi_x + \varphi\varphi_y = 0$. To be precise, the arguments of φ, $1 - 1/\varphi$ and $1/(1 - \varphi)$ are proportional to the densities of tiles of the three orientations. These densities extend continuously to constants on the frozen regions. For polygonal domains like the one in Figure 1, the solution to this free boundary problem is algebraic. In particular, it turns out that the boundary of the frozen region in the figure is an inscribed cardioid (a degree 4, genus 0 curve with 1 real and 2 complex cusps — the yellow curve in the figure).

Variant 1

The set of all edges of a triangle is a dimer covering of it.	**a)** true **b)** false
The dimer model does not provide an accurate description of an important physical phenomenon.	**a)** true **b)** false
All entries of the adjacency matrix of a graph are equal to 0 or 1.	**a)** true **b)** false
The height function of a random tiling with many small rhombic tiles looks somewhat regular.	**a)** true **b)** false
The yellow curve in Figure 1 is a cardioid.	**a)** true **b)** false

Variant 1

№ 5 Fill in the gaps in the text with the suitable words:

Cauchy Integral

(from the Springer Online Encyclopedia of Mathematics, URL: http://www.encyclopediaofmath.org)

A Cauchy integral is an integral with the Cauchy kernel, $\frac{1}{2\pi i(\zeta - z)}$, expressing the **__1__** of a regular analytic function $f(z)$ in the interior of a contour L in terms of its values on L. More precisely: Let $f(z)$ be a regular analytic function of the complex **__2__** *z* in a domain *D* and let *L* be a closed piecewise-smooth Jordan **__3__** lying in *D* together with its interior *G*; it is assumed that *L* is described in the counter-clockwise sense. Then one has the following formula, which is of fundamental importance in the theory of analytic **__4__** of one complex variable and which is known as the Cauchy integral **__5__**:

$$f(z)=\frac{1}{2\pi i}\int_L \frac{f(\zeta)d\zeta}{\zeta-z}$$

The integral on the right of this formula is also called a Cauchy integral.

1 A. values	**B.** packages	**C.** numbers	**D.** nodes
2 A. constant	**B.** letter	**C.** variable	**D.** solution
3 A. vector	**B.** direction	**C.** annulus	**D.** curve
4 A. operators	**B.** functions	**C.** equations	**D.** arguments
5 A. formula	**B.** theorem	**C.** proposition	**D.** claim

Variant 1

№ 6 Read this extract from a linear algebra text and complete the sentences with the correct options A – D

Let *A* **__1__** a linear operator acting on a Euclidean vector space *V*. Suppose that *A* **__2__** symmetric. Then all **__3__** eigenvalues are real. Moreover, eigenvectors corresponding **__4__** different eigenvalues are orthogonal, **__5__** linearly independent. We define the multiplicity of an eigenvalue **__6__** the dimension of the corresponding eigenspace. If all eigenvalues are simple, i.e. have multiplicity one, **__7__** the eigenvectors are **__8__** proportional or orthogonal to each other. **__9__** an orthonormal basis in every eigenspace of *A*, one can form an orthonormal basis in *V* consisting **__10__** eigenvectors, even if *A* has multiple eigenvalues.

1	**A**	*is*	**B**	*be*	**C**	*to be*	**D**	*was*
2	**A**	*is*	**B**	*be*	**C**	*to be*	**D**	*was*
3	**A**	*it*	**B**	*its*	**C**	*it's*	**D**	*it is*
4	**A**	*with*	**B**	*from*	**C**	*to*	**D**	*at*
5	**A**	*hence*	**B**	*provided that*	**C**	*because*	**D**	*due to*
6	**A**	*for*	**B**	*like*	**C**	*as*	**D**	*from*
7	**A**	*then*	**B**	*than*	**C**	*when*	**D**	*where*
8	**A**	*or*	**B**	*either*	**C**	*neither*	**D**	*nor*
9	**A**	*Choosing*	**B**	*Chosen*	**C**	*To choose*	**D**	*Chose*
10	**A**	*from*	**B**	*with*	**C**	*of*	**D**	*about*